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# Convergence of Common Fixed Point for Asymptotically Quasi-Nonexpansive Mappings in Convex Metric Spaces

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Abstract. In this paper, the necessary and sufficient conditions for three-step iterative sequences with errors to converge to a common fixed point for three asymptotically quasi-nonexpansive mappings is established in convex metric spaces. The results of this paper are generalizations and improvements of the corresponding results of Chang [1] - [3], Kim et al. [8], Liu [9] - [11], Ghosh and Debnath [4], Xu and Noor [15], Shahzad and Udomene [13], Khan and Takahashi [6] and Khan and Ud-din [7].

## 1. Introduction and Preliminaries

Throughout this paper, we assume that $E$ is a metric space, $F(T)$ and $D(T)$ are the set of fixed points and domain of $T$ respectively and $\mathbb{N}$ is the set of all positive integers.

**Definition 1.1 ( [8]).** Let $T\colon D(T) \subset E \to E$ be a mapping.

(1) The mapping $T$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in D(T).$$

(2) The mapping $T$ is said to be quasi-nonexpansive if

$$d(Tx, p) \leq d(x, p), \quad \forall x \in D(T),\ \forall p \in F(T).$$

(3) The mapping $T$ is said to be asymptotically nonexpansive if there exists a sequence $r_n \in [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + r_n) d(x, y), \quad \forall x, y \in D(T),\ \forall n \in \mathbb{N}.$$

(4) The mapping $T$ is said to be asymptotically quasi-nonexpansive if there exists a sequence $r_n \in [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, p) \leq (1 + r_n) d(x, p), \quad \forall x \in D(T),\ \forall p \in F(T),\ \forall n \in \mathbb{N}.$$

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**Remark 1.1.** From the definition 1.1, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold.

The iterative approximation problems of fixed points for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings in Hilbert spaces or Banach spaces have been studied extensively by many others. In 1973, Petryshyn and Williamson [12] obtained a necessary and sufficient condition for Picard iterative sequences and Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings and later, the result of [12] was extended by Ghosh and Debnath [4] to Ishikawa iterative sequences. Recently, Chang [1] - [3] has proved some other kinds of necessary and sufficient conditions for Ishikawa iterative sequences with errors to converge to a fixed point for asymptotically nonexpansive mappings and Xu and Noor [15] have established a convergence theorem of three-step iterative sequences with errors for asymptotically nonexpansive mappings in uniformly convex Banach spaces. In particular, Liu [9] obtained a necessary and sufficient condition for Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings in Banach spaces to converge to a fixed point and he [10] has also extended his result [9] to Ishikawa iterative sequences with errors. Furthermore, Kim et al. [8] extended the result of Liu [9] to modified three-step iterative sequences with mixed errors.

In 2001, Khan and Takahashi [6] have established a convergence theorem for two asymptotically nonexpansive mappings and in 2004, Khan and Ud-din [7] have established a convergence theorem for a scheme with errors for two nonexpansive mappings. In 2006, Shahzad and Udomene [13] extended the corresponding result of [6] and [7] for two asymptotically quasi-nonexpansive mappings and also established a necessary and sufficient condition for convergence of two asymptotically quasi-nonexpansive mappings in Banach spaces.

The purpose of this paper is to study some necessary and sufficient conditions for three-step iterative sequences with errors to converge to common fixed points for three asymptotically quasi-nonexpansive mappings in convex metric spaces. The results of this paper are generalization and improvements of the corresponding results in Chang [1] - [3], Ghosh and Debnath [4], Ud-din and Khan [5], Khan and Takahashi [6], Khan and Ud-din [7], Kim et al. [8], Liu [9] - [11], Shahzad and Udomene [13] and Xu and Noor [15].

For the sake of convenience, we first recall some definitions and notations.

**Definition 1.2.** Let $(E, d)$ be a metric space and $I = [0, 1]$. A mapping $W\colon E^3 \times I^3 \to E$ is said to be a convex structure on $E$ if it satisfies the following conditions: for all $u, x, y, z \in E$ and for all $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$,

(1) $W(x, y, z; \alpha, 0, 0) = x$,
(2) $d(u, W(x, y, z; \alpha, \beta, \gamma)) \leq \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z)$.

If $(E, d)$ is a metric space with a convex structure $W$, then $(E, d)$ is called a *convex metric space* and denotes it by $(E, d, W)$.

**Remark 1.2.** Every linear normed space is a convex metric space, where a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [14]).

**Definition 1.3.** (1) Let $(E, d, W)$ be a convex metric space, $T_1, T_2, T_3\colon E \to E$ be mappings and let $x_1 \in E$ be a given point. Then the sequence $\{x_n\}$ defined by

$$
\begin{aligned}
x_{n+1} &= W(x_n, T_1^n y_n, u_n; a_n, b_n, c_n), \\
y_n &= W(x_n, T_2^n z_n, v_n; a'_n, b'_n, c'_n), \\
z_n &= W(x_n, T_3^n x_n, w_n; a''_n, b''_n, c''_n), \quad \forall n \in \mathbb{N},
\end{aligned}
\tag{1.1}
$$

is called the three-step iterative sequence with errors for three mappings $T_1, T_2, T_3$, where $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$, $\{c_n\}$, $\{c'_n\}$ and $\{c''_n\}$ are nine sequences in $[0, 1]$ satisfying the following conditions:

$a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$, $\forall n \in \mathbb{N}$,

and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are three bounded sequences in $E$.

(2) In (1.1), if $b''_n = c''_n = 0$, for all $n = 1, 2, \dots$, then $z_n = x_n$ and $T_1 = T_2 = T_3 = T$. Then the sequence $\{x_n\}$ defined by

$$
\begin{aligned}
x_{n+1} &= W(x_n, T^n y_n, u_n; a_n, b_n, c_n), \\
y_n &= W(x_n, T^n x_n, v_n; a'_n, b'_n, c'_n), \quad \forall n \in \mathbb{N},
\end{aligned}
\tag{1.2}
$$

is called the Ishikawa type (or two-step) iterative sequence with errors for the mapping $T$, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are six sequences in $[0, 1]$ satisfying the conditions:

$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, $\forall n \in \mathbb{N}$,

and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in $E$.

## 2. Main Results

In order to prove our main result, we will first prove the following lemma.

**Lemma 2.1.** Let $(E, d, W)$ be a convex metric space, $T_1, T_2, T_3\colon E \to E$ be three asymptotically quasi-nonexpansive mappings satisfying $\sum_{n=1}^{\infty} r_n < \infty$ where $\{r_n\}$ is the sequence appeared in Definition 1.1, and $F = \bigcap_{i=1}^{3} F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative sequences with errors defined by (1.1). Then

(**a**) $d(x_{n+1}, p) \leq (1 + r_n)^3 d(x_n, p) + B_n$, $\forall p \in F$, $n \in \mathbb{N}$,
where $B_n = A_n b_n(1 + r_n) + c_n d(u_n, p)$, $A_n = b'_n c''_n(1 + r_n)d(w_n, p) + c'_n d(v_n, p)$
and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are three bounded sequences in $E$.

(**b**) there exists a constant $M > 0$ such that
$d(x_m, p) \leq M.d(x_n, p) + M.\sum_{j=n}^{m-1} B_j$, $\forall p \in F$, $m > n$.

*Proof.* (**a**) Let $p \in F = \bigcap_{i=1}^{3} F(T_i)$. Since $T_i (i = 1, 2, 3)$ is asymptotically quasi-nonexpansive, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T_1^n y_n, u_n; a_n, b_n, c_n), p) \\ &\leq a_n d(x_n, p) + b_n d(T_1^n y_n, p) + c_n d(u_n, p) \\ &\leq a_n d(x_n, p) + b_n(1 + r_n)d(y_n, p) + c_n d(u_n, p) \end{aligned} \tag{2.1}$$

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T_2^n z_n, v_n; a'_n, b'_n, c'_n), p) \\ &\leq a'_n d(x_n, p) + b'_n d(T_2^n z_n, p) + c'_n d(v_n, p) \\ &\leq a'_n d(x_n, p) + b'_n(1 + r_n)d(z_n, p) + c'_n d(v_n, p) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(z_n, p) &= d(W(x_n, T_3^n x_n, w_n; a''_n, b''_n, c''_n), p) \\ &\leq a''_n d(x_n, p) + b''_n d(T_3^n x_n, p) + c''_n d(w_n, p) \\ &\leq a''_n d(x_n, p) + b''_n(1 + r_n)d(x_n, p) + c''_n d(w_n, p) \\ &\leq a''_n(1 + r_n)d(x_n, p) + b''_n(1 + r_n)d(x_n, p) + c''_n d(w_n, p) \\ &\leq (a''_n + b''_n)(1 + r_n)d(x_n, p) + c''_n d(w_n, p) \\ &= (1 - c''_n)(1 + r_n)d(x_n, p) + c''_n d(w_n, p) \\ &\leq (1 + r_n)d(x_n, p) + c''_n d(w_n, p) \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we have

(2.4)

$$
\begin{aligned}
d(y_n, p) &\leq a'_n d(x_n, p) + b'_n(1 + r_n)[(1 + r_n)d(x_n, p) + c''_n d(w_n, p)] \\
&\quad + c'_n d(v_n, p) \\
&\leq a'_n d(x_n, p) + b'_n(1 + r_n)^2 d(x_n, p) + b'_n(1 + r_n)c''_n d(w_n, p) \\
&\quad + c'_n d(v_n, p) \\
&\leq a'_n d(x_n, p) + b'_n(1 + r_n)^2 d(x_n, p) + b'_n(1 + r_n)c''_n d(w_n, p) \\
&\quad + c'_n d(v_n, p) \\
&\leq a'_n(1 + r_n)^2 d(x_n, p) + b'_n(1 + r_n)^2 d(x_n, p) + b'_n(1 + r_n)c''_n d(w_n, p) \\
&\quad + c'_n d(v_n, p) \\
&\leq (a'_n + b'_n)(1 + r_n)^2 d(x_n, p) + b'_n(1 + r_n)c''_n d(w_n, p) + c'_n d(v_n, p) \\
&= (1 - c'_n)(1 + r_n)^2 d(x_n, p) + b'_n(1 + r_n)c''_n d(w_n, p) + c'_n d(v_n, p) \\
&\leq (1 + r_n)^2 d(x_n, p) + A_n
\end{aligned}
$$

where $A_n = b'_n(1+r_n)c''_n d(w_n, p) + c'_n d(v_n, p)$. And again, substituting (2.4) into (2.1), it follows that

$$
\begin{aligned}
d(x_{n+1}, p) &\leq a_n d(x_n, p) + b_n(1 + r_n)[(1 + r_n)^2 d(x_n, p) + A_n] \\
&\quad + c_n d(u_n, p) \\
&\leq a_n d(x_n, p) + b_n(1 + r_n)^3 d(x_n, p) + b_n(1 + r_n)A_n + c_n d(u_n, p) \\
&\leq a_n(1 + r_n)^3 d(x_n, p) + b_n(1 + r_n)^3 d(x_n, p) + b_n(1 + r_n)A_n + c_n d(u_n, p) \\
&\leq (a_n + b_n)(1 + r_n)^3 d(x_n, p) + b_n(1 + r_n)A_n + c_n d(u_n, p) \\
&= (1 - c_n)(1 + r_n)^3 d(x_n, p) + b_n(1 + r_n)A_n + c_n d(u_n, p) \\
&\leq (1 + r_n)^3 d(x_n, p) + B_n
\end{aligned}
$$

where $B_n = b_n(1 + r_n)A_n + c_n d(u_n, p)$. This completes the proof of (a).

(**b**) If $x \geq 0$, then $1 + x \leq e^x$ and $(1+x)^3 \leq e^{3x}$. Therefore from (a) we can obtain that

$$\begin{aligned}
d(x_m, p) &\leq (1 + r_{m-1})^3 d(x_{m-1}, p) + B_{m-1} \\
&\leq e^{3r_{m-1}} d(x_{m-1}, p) + B_{m-1} \\
&\leq e^{3r_{m-1}} [e^{3r_{m-2}} d(x_{m-2}, p) + B_{m-2}] + B_{m-1} \\
&\leq e^{3(r_{m-1}+r_{m-2})} d(x_{m-2}, p) + e^{3r_{m-1}} B_{m-2} + B_{m-1} \\
&\leq e^{3(r_{m-1}+r_{m-2})} d(x_{m-2}, p) + e^{3r_{m-1}} [B_{m-1} + B_{m-2}] \\
&\leq \dots\dots \\
&\leq \dots\dots \\
&\leq e^{3(r_{m-1}+r_{m-2}+\cdots+r_n)} d(x_n, p) \\
&+ e^{3(r_{m-1}+r_{m-2}+\cdots+r_n)} [B_{m-1} + B_{m-2} + \cdots + B_n] \\
&\leq e^{3\sum_{j=n}^{m-1} r_j} d(x_n, p) + e^{3\sum_{j=n}^{m-1} r_j} . \sum_{j=n}^{m-1} B_j \\
&\leq M.d(x_n, p) + M. \sum_{j=n}^{m-1} B_j,
\end{aligned}$$

where $M = e^{3\sum_{j=n}^{m-1} r_j}$. This completes the proof of (b). □

**Lemma 2.1** [10]. Let the number of sequences $\{a_n\}$, $\{b_n\}$ and $\{\lambda_n\}$ satisfy that $a_n \geq 0$, $b_n \geq 0$, $\lambda_n \geq 0$, $a_{n+1} \leq (1 + \lambda_n)a_n + b_n$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then

(a) $\lim_{n\to\infty} a_n$ exists.

(b) If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Now, we are in a position to prove the main results. $D_d(y, S)$ denotes the distance from $y$ to set $S$, that is, $D_d(y, S) = \inf\{d(y, s) : s \in S\}$.

**Theorem 2.1.** Let $(E, d, W)$ be a complete convex metric space, $T_1, T_2, T_3 : E \to E$ be three asymptotically quasi-nonexpansive mappings and $F = \bigcap_{i=1}^{3} F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative sequence with errors defined by (1.1) and $\{r_n\}$, $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$ be four sequences satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} r_n < \infty$,

(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$, $\sum_{n=1}^{\infty} c''_n < \infty$,

where $\{r_n\}$ is a sequence appeared in Definition 1.1 and $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$ are three sequences appeared in (1.1). Then the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, 3\}$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, F) = 0.$$

*Proof.* The necessity is obvious. Now, we prove the sufficiency. Suppose that the condition $\liminf_{n\to\infty} D_d(x_n, F) = 0$ is satisfied. Then from Lemma

2.1(a), we have

$$d(x_{n+1},p) \le (1+r_n)^3 d(x_n,p) + B_n, \quad \forall p \in F, \quad \forall n \in \mathbb{N}, \tag{2.5}$$

where $B_n = b_n(1+r_n)A_n + c_n d(u_n,p)$ and $A_n = b'_n(1+r_n)c''_n d(w_n,p) + c'_n d(v_n,p)$. Since $0 \le b_n, b'_n \le 1$, $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$, $\sum_{n=1}^{\infty} c''_n < \infty$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are three bounded sequences, we have $\sum_{n=1}^{\infty} A_n < \infty$ and so $\sum_{n=1}^{\infty} B_n < \infty$. From (2.5) we can obtain that

$$D_d(x_{n+1},F) \le (1+r_n)^3 D_d(x_n,F) + B_n.$$

Since $\liminf_{n\to\infty} D_d(x_n,F) = 0$, by Lemma 2.2, we have

$$\lim_{n\to\infty} D_d(x_n,F) = 0.$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. By Lemma 2.1(b), there exists a constant $M > 0$ such that

$$d(x_m,p) \le M \cdot d(x_n,p) + M. \sum_{j=n}^{m-1} B_j, \ \forall p \in F, \ m > n. \tag{2.6}$$

Since $\lim_{n\to\infty} D_d(x_n,F) = 0$ and $\sum_{n=1}^{\infty} B_n < \infty$, there exists a constant $N_1$ such that for all $n \ge N_1$,

$$D_d(x_n,F) < \frac{\varepsilon}{4M} \ \ \text{and} \ \ \sum_{j=N_1}^{\infty} B_j < \frac{\varepsilon}{6M}.$$

We note that there exists $p_1 \in F$ such that $d(x_{N_1},p_1) < \frac{\varepsilon}{3M}$. It follows that from (2.6) that for all $m > n > N_1$,

$$\begin{aligned} d(x_m,x_n) &\le d(x_m,p_1) + d(x_n,p_1) \\ &\le M \cdot d(x_{N_1},p_1) + M \sum_{j=N_1}^{m-1} B_j + M d(x_{N_1},p_1) + M \sum_{j=N_1}^{n-1} B_j \\ &< M \cdot \frac{\varepsilon}{3M} + M\frac{\varepsilon}{6M} + M\frac{\varepsilon}{3M} + M\frac{\varepsilon}{6M} \\ &= \varepsilon. \end{aligned} \tag{2.7}$$

Since $\varepsilon$ is an arbitrary positive number, (2.7) implies that $\{x_n\}$ is a Cauchy sequence. From the completeness of this, $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = p$. It will be proven that $p$ is a common fixed point. Let $\bar{\varepsilon} > 0$. Since $\lim_{n\to\infty} x_n = p$, there exists a natural number $N_2$ such that for all $n \ge N_2$,

$$d(x_n,p) < \frac{\bar{\varepsilon}}{2(2+r_1)}. \tag{2.8}$$

$\lim_{n\to\infty} D_d(x_n,F) = 0$ implies that there exists a natural number $N_3 \ge N_2$ such that for all $n \ge N_3$,

$$D_d(x_n,F) < \frac{\bar{\varepsilon}}{3(4+3r_1)}. \tag{2.9}$$

Therefore, there exists a $p^* \in F$ such that

$$d(x_{N_3}, p^*) < \frac{\bar{\varepsilon}}{2(4+3r_1)}. \tag{2.10}$$

From (2.8) and (2.10), we have for any $i \in I$

$$\begin{aligned} d(T_ip, p) &\leq d(T_ip, p^*) + d(p^*, T_ix_{N_3}) + d(T_ix_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\ &= d(T_ip, p^*) + 2d(T_ix_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\ &\leq (1+r_1)d(p, p^*) + 2(1+r_1)d(x_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\ &\leq (1+r_1)[d(p, x_{N_3}) + d(x_{N_3}, p^*)] + 2(1+r_1)d(x_{N_3}, p^*) \\ &\quad + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\ &= (2+r_1)d(x_{N_3}, p) + (4+3r_1)d(x_{N_3}, p^*) \\ &< (2+r_1).\frac{\bar{\varepsilon}}{2(2+r_1)} + (4+3r_1).\frac{\bar{\varepsilon}}{2(4+3r_1)} \\ &= \bar{\varepsilon}. \end{aligned}$$

Since $\bar{\varepsilon}$ is an arbitrary positive number, this implies that $T_ip = p$. Hence $p \in F(T_i)$ for all $i \in I$ and so $p \in F = \bigcap_{i=1}^{3} F(T_i)$. Thus the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, 3\}$. This completes the proof. □

In (1.1), if $T_1 = T_2 = T_3 = T$, $b''_n = c''_n = 0$ for all $n = 1, 2, \ldots$, then $z_n = x_n$. Therefore, the following corollary can be obtained from Theorem 2.1 immediately.

**Corollary 2.1.** Let $(E, d, W)$ be a complete convex metric space, $T\colon E \to E$ be an asymptotically quasi-nonexpansive mapping and $F(T)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the Ishikawa type iterative sequence with errors defined by (1.2) and $\{r_n\}$, $\{c_n\}$, $\{c'_n\}$ be three sequences satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} r_n < \infty$,
(ii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$,

where $\{r_n\}$ is a sequence appeared in Definition 1.1 and $\{c_n\}$, $\{c'_n\}$ are two sequences appeared in (1.2). Then the iterative sequence $\{x_n\}$ converges to a fixed point of $T$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, F(T)) = 0.$$

By using the same method in Theorem 2.1, we can easily obtain the following theorem.

**Theorem 2.2.** Let $(E, d, W)$ be a complete convex metric space, $T_1, T_2, T_3\colon E \to E$ be three quasi-nonexpansive mappings and $F = \bigcap_{i=1}^{3} F(T_i)$ be a nonempty set. For a given $x_1 \in E$, let $\{x_n\}$ be the three-step iterative sequence with errors defined by:

$$(2.11)\quad \begin{aligned} x_{n+1} &= W(x_n, T_1 y_n, u_n; a_n, b_n, c_n),\\ y_n &= W(x_n, T_2 z_n, v_n; a'_n, b'_n, c'_n),\\ z_n &= W(x_n, T_3 x_n, w_n; a''_n, b''_n, c''_n), \quad \forall n \in \mathbb{N}, \end{aligned}$$

and $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$ are three sequences satisfying the following condition:
(i) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$, $\sum_{n=1}^{\infty} c''_n < \infty$,

where $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$ are three sequences appeared in (1.3). Then the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, 3\}$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, F) = 0.$$

From Theorem 2.1, we can also obtain the following result for the Banach space.

**Theorem 2.3.** Let $E$ be a real Banach space, $T_1, T_2, T_3 \colon E \to E$ be three asymptotically quasi-nonexpansive mappings satisfying the condition (i) in Theorem 2.1 and $F = \bigcap_{i=1}^{3} F(T_i)$ be a nonempty set. Let $\{x_n\}$ be the three-step iterative sequence with errors defined by

$$\begin{aligned} &x_1 \in E,\\ x_{n+1} &= a_n x_n + b_n T_1^n y_n + c_n u_n,\\ y_n &= a'_n x_n + b'_n T_2^n z_n + c'_n v_n,\\ z_n &= a''_n x_n + b''_n T_3^n x_n + c''_n w_n, \ \forall n \in \mathbb{N}, \end{aligned}$$

where $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are three bounded sequences in $E$ and $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$, $\{c_n\}$, $\{c'_n\}$ and $\{c''_n\}$ are nine sequences in $[0, 1]$ satisfying $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$, $\forall n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$, $\sum_{n=1}^{\infty} c''_n < \infty$. Then the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, 3\}$ if and only if

$$\liminf_{n\to\infty} D(x_n, F) = 0.$$

where $D_d(y, S) = \inf\{d(y, s) : s \in S\}$.

*Proof.* Since $E$ is a Banach space, it is a complete convex metric space with a convex structure $W(x, y, z : \alpha, \beta, \gamma) := \alpha x + \beta y + \gamma z$, for all $x, y, z \in E$ and for all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$. Therefore, the conclusion of Theorem 2.3 can be obtained from Theorem 2.1 immediately. □

**Remark 2.1.** (1) Theorem 2.1 and 2.2 are two new convergence theorems of three-step iterative sequences with errors for nonlinear mappings in convex metric spaces. These two theorems generalize and improves the corresponding results of [9]- [11], [1]- [3] and [4, 6, 7, 12, 13, 15].

(2) Theorem 2.3 generalizes and improves the corresponding results of Kim et al. [8], Liu [10, 11], Shahzad and Udomene [13], Khan and Takahashi [6], Khan and Ud-din [7] and Xu and Noor [15].

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